APPENDIX 2: Mathematical Appendix for Spending Change Calculations

Suppose we assign variables as follows, for Period 1 (before a change in the local tax rate) and Period 2 (after a change in the local tax rate):

\[
\begin{align*}
A_i & = \text{Allocation in each Period} \\
E_i & = \text{Taxable expenditures in each Period} \\
S_i & = \text{Total spending in each Period} \\
T_i & = \text{Tax rate in each Period} \\
\Delta_T & = \text{Percent change in the tax rate from Period 1 to Period 2} \\
\Delta_A & = \text{Percent change in Arlington tax revenues from Period 1 to Period 2} \\
\Delta_E & = \text{Percent change in taxable expenditures from Period 1 to Period 2} \\
\Delta_S & = \text{Percent change in total spending from Period 1 to Period 2} \\
i & = 1, 2
\end{align*}
\]

We can begin by defining \(A_1, E_1, A_2, E_2, S_1, \) and \(S_2\) simply as:

\[
\begin{align*}
A_1 & = T_1E_1 \text{ and } A_2 = T_2E_2, \\
E_1 & = \frac{A_1}{T_1} \text{ and } E_2 = \frac{A_2}{T_2}, \\
S_1 & = (1 + T_1)E_1 \text{ and } S_2 = (1 + T_2)E_2.
\end{align*}
\]

Additionally, we define \(\Delta_T, \Delta_A, \Delta_E, \) and \(\Delta_S\) as:

\[
\begin{align*}
\Delta_T & = \frac{T_2}{T_1} - 1, \Delta_A = \frac{A_2}{A_1} - 1, \Delta_E = \frac{E_2}{E_1} - 1, \text{ and } \Delta_S = \frac{S_2}{S_1} - 1.
\end{align*}
\]

Then it follows naturally from the form of the regression model that our percentage change for the tax increment in question will be

\[
\Delta_A = (\Delta_T + 1)^\beta - 1 = \left(\frac{T_2}{T_1}\right)^\beta - 1.
\]

From here, we can substitute in order to calculate the analogous percent changes in both taxable expenditures and total spending in Periods 1 and 2, beginning with \(A_2:\)

\[
A_2 = (1 + \Delta_A)A_1 = \left(\frac{T_2}{T_1}\right)^\beta T_1E_1,
\]

by combining the fact that \(A_1 = T_1E_1\) and \(\Delta_A = \left(\frac{T_2}{T_1}\right)^\beta - 1.\) from above.
Now, we can represent $\Delta_E$ in terms of only our elasticity from the regression model and the analogous tax rates in the two periods, beginning with substitution into $E_2$ from above:

$$E_2 = \frac{A_2}{T_2} = \frac{(T_2/T_1)^\beta T_1E_1}{T_2},$$

Substituting this into the definition for $\Delta_E$,

$$\Delta_E = \frac{E_2}{E_1} - 1 = \frac{(T_2/T_1)^\beta T_1E_1}{T_2E_1} - 1,$$

and cancelling $E_1$ from the top and bottom,

$$\Delta_E = \frac{(T_2/T_1)^\beta}{T_2} T_1 - 1 = \left(\frac{T_1}{T_2}\right) \left(\frac{T_2}{T_1}\right)^\beta - 1$$

$$= \left(\frac{T_1}{T_2}\right) \left(\frac{T_2}{T_1}\right)^{\beta-1} - 1$$

$$= \left(\frac{T_2}{T_1}\right)^{\beta-1} - 1.$$

Lastly, we can evaluate a simple expression of known values for the change in total spending in the area across the two periods:

$$\Delta_S = \frac{(1 + T_2)E_2}{(1 + T_1)E_1} - 1, \text{ by substitution from } S_i = (1 + T_i)E_i,$$

$$\frac{E_2}{E_1} = \Delta_E + 1, \text{ by rearrangement from the definition } \Delta_E = \frac{E_2}{E_1} - 1,$$

$$\Delta_S = \frac{(1 + T_2)}{(1 + T_1)} (\Delta_E + 1) - 1, \text{ by substitution.}$$
We simplify this to:

\[
\Delta_S = (1 + T_2) \left[ \left( \frac{T_2}{T_1} \right)^{\beta - 1} - 1 \right] + 1 - 1 \\
= \frac{(1 + T_2)(T_2)^{\beta - 1}}{(1 + T_1)(T_1)} - 1
\]

Once the regression model is estimated, the elasticity for the local tax rate can be substituted for \( \beta \), and the known tax rates in the two periods are substituted for \( T_1 \) and \( T_2 \). For the change from a 1.0\% to a 1.5\% local tax rate:

\[
\beta = 0.893, \\
T_1 = 0.010 \\
T_2 = 0.015
\]

\[
\Delta_A = \left( \frac{T_2}{T_1} \right)^{\beta} - 1 = \left( \frac{0.015}{0.010} \right)^{0.893} - 1 \approx 0.4363 = 43.63\%
\]

\[
\Delta_E = \left( \frac{T_2}{T_1} \right)^{\beta - 1} - 1 = \left( \frac{0.015}{0.010} \right)^{0.893 - 1} - 1 \approx -0.0425 = -4.25\%
\]

\[
\Delta_S = \frac{(1 + T_2)(T_2)^{\beta - 1}}{(1 + T_1)(T_1)} - 1 = \frac{(1 + 0.015)(0.015)^{0.893 - 1}}{(1 + 0.010)(0.010)} - 1 \approx -0.0377 = -3.77\%
\]

Similarly, for the change in the local tax rate from 1.25\% to 1.75\%:

\[
\beta = 0.893, \\
T_1 = 0.0125 \\
T_2 = 0.0175
\]

\[
\Delta_A = \left( \frac{T_2}{T_1} \right)^{\beta} - 1 = \left( \frac{0.0175}{0.0125} \right)^{0.893} - 1 \approx 0.3505 = 35.05\%
\]

\[
\Delta_E = \left( \frac{T_2}{T_1} \right)^{\beta - 1} - 1 = \left( \frac{0.0175}{0.0125} \right)^{0.893 - 1} - 1 \approx -0.0354 = -3.54\%
\]

\[
\Delta_S = \frac{(1 + T_2)(T_2)^{\beta - 1}}{(1 + T_1)(T_1)} - 1 = \frac{(1 + 0.0175)(0.0175)^{0.893 - 1}}{(1 + 0.0125)(0.0125)} - 1 \approx -0.0306 = -3.06\%.
\]